SOME SUBORDINATION PROPERTIES FOR A SUBCLASS OF SPIRALLIKE FUNCTIONS OF TYPE $\beta^*$

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Abstract

In this paper, we investigate several subordination properties or a subclass of spirallike of type $\beta$ denoted by $S^\beta_\alpha$, which was introduced by R. J. Libera. The results presented here would generalize recent work of Y. C. Kim and H. M. Srivastava.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

We denote by $S$ the class of all functions in $A$ which are univalent in $U$.

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A function $f \in \mathcal{S}$ is said to be starlike in $\mathbb{U}$ if the image $f(\mathbb{U})$ is starlike with respect to 0. It is well known that a function $f$ is starlike if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the class of all functions in $\mathcal{S}$ which are starlike in $\mathbb{U}$. Obviously, $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}$ holds.

A function $f \in \mathcal{S}$ is said to be convex in $\mathbb{U}$ if the image $f(\mathbb{U})$ is convex. The function $f$ is convex in $\mathbb{U}$ if and only if

$$\Re \left( 1 + \frac{zf''(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{K}$ the class of all function in $\mathcal{S}$ which are convex in $\mathbb{U}$ (see for details [1] and [11]). Obviously, $\mathcal{K} \subset \mathcal{S} \subset \mathcal{A}$ holds.

Let $a$, $b$ and $c$ be complex numbers with $c \neq 0, -1, -2, \ldots$. Then the Gaussian hypergeometric function $_2F_1$ is defined by

$$_2F_1(z) \equiv _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} = \{1, 2, \ldots\}). \end{cases}$$

We note that the Gaussian hypergeometric $_2F_1$ series converges absolutely for $z \in \mathbb{U}$.

For analytic functions $g$ and $h$ in $\mathbb{U}$, $g$ is said to be subordinate to $h$ if there exists an analytic function $\omega$ such that (see for example [8])

$$\omega(0) = 0, \quad |\omega(z)| < 1 \quad \text{and} \quad g(z) = h(\omega(z)) \quad (z \in \mathbb{U}).$$

This subordination will be denoted here by

$$g \prec h \quad (z \in \mathbb{U}),$$
or, conventionally, by
\[ g(z) \prec h(z) \ (z \in U). \]

In particular, when \( h \) is univalent in \( U \),
\[ g \prec h \iff g(0) = h(0) \text{ and } g(U) \subset h(U) \ (z \in U). \]

**Definition 1.** (see [9]). Let \( f \in \mathcal{A}, \, \alpha \in [0, 1) \), and
\[ \Re \left[ \frac{zf'(z)}{f(z)} \right] > \alpha \quad (z \in U). \] (6)

We say that \( f \) is a starlike function of order \( \alpha \). Let \( S^\ast (\alpha) \) denote the whole starlike functions of order \( \alpha \) on \( U \).

Spaček [12] extended the class of \( S^\ast \), and obtained the class of spirallike function of type \( \beta \). In the same article, the author gave an analytical characterization of spirallikeness of type \( \beta \) on \( U \).

**Definition 2.** (see [12]). Let \( f \in \mathcal{A} \) and \( \beta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), then \( f(z) \) is a spirallike function of type \( \beta \) on \( U \) if and only if
\[ \Re \left[ e^{i\beta} \frac{zf'(z)}{f(z)} \right] > 0 \quad (z \in U). \] (7)

We denote the whole spirallike functions of type of \( \beta \) on \( U \) by \( \hat{S}_\beta \).

In [7], Liber extended the classes \( S^\ast (\alpha) \) and \( \hat{S}_\beta \) by introducing the following analytic functions class \( \hat{S}_\alpha^\beta \) on \( U \).

**Definition 3.** (see [2]). Let \( f \in \mathcal{A}, \, \alpha \in [0, 1) \) and \( \beta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \), then \( f \in \hat{S}_\alpha^\beta \) if and only if
\[ \Re \left[ e^{i\beta} \frac{zf'(z)}{f(z)} \right] > \alpha \cos \beta, \quad (z \in U). \] (8)

Obviously, when \( \beta = 0 \), \( f \in S^\ast (\alpha) \); while \( \alpha = 0 \), \( f \in \hat{S}_\beta \).
Recently, Xu and Lu [13] proved that \( \hat{S}_\alpha^\beta \) if and only if there is a starlike function \( s(z) \) in \( S^* \) such that
\[
\frac{f(z)}{z} = \left( \frac{s(z)}{z} \right)^{\gamma(1-\alpha)} \quad (z \in \mathbb{U}),
\]
where \( \gamma = e^{-i\beta} \cos \beta \). With this \( \gamma \), we let \( g \) be the special function obtained by choosing the koebe function \( s(z) \) above, that is,
\[
g(z) = \frac{z}{(1-z)^{2\gamma(1-\alpha)}}, \quad \gamma = e^{-i\beta} \cos \beta \quad (z \in \mathbb{U}).
\]

The function \( g(z) \) plays the role the Koebe function in extremal subordination problems for the class \( \hat{S}_\alpha^\beta \).

The integral transformation \( J \), defined by
\[
J(f)(z) = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(\zeta)}{\zeta} d\zeta
\]
is called the Alexander transformation and it was introduced by Alexander (see [3] and [4]). Alexander showed that the integral transformation \( J \) maps the class \( S^* \) of starlike functions onto the class \( \mathcal{K} \) of convex functions in a one-to-one fashion. In 1960, Biernacki conjectured that \( J(S) \subset S \). Subsequently, Krzyz and Lewandowski disproved it in 1963 by giving the example \( f(z) = z(1-iz)^{i-1} \), which is a spirallike function of type \( \frac{\pi}{4} \) but is transformed into a non-univalent function by \( J \) (see [1]).

In [5], Y. C. Kim and T. Sugawa showed that the Alexander transform of a \( \beta \)-spirallike function is univalent when
\[
\cos \beta \leq \frac{1}{2},
\]
which settles the problem posed earlier by M. S. Robertson [10].

Recently, Xu and Lu [13] proved that the Alexander transform of a function \( f(z) \) in the class \( \hat{S}_\alpha^\beta \) is univalent when
which generalized the work of Y. C. Kim and T. Sugawa [5].

In this paper, inspired by [6], we investigate several subordination properties for the class $\hat{S}^\beta_\alpha$. These results extend the corresponding results showed by Y. C. Kim and H. M. Srivastava [6].

2. Main Subordination Properties

In this section, we state and prove our subordination properties involving the function class $\hat{S}^\beta_\alpha$ given by Definition 3.

Theorem 1. Let $\alpha \in [0, 1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and suppose that $\gamma = e^{-\beta} \cos \beta$. If $f \in \hat{S}^\beta_\alpha$, then

$$\frac{zf'(z)}{f(z)} < \frac{zg'(z)}{g(z)} \quad (z \in \mathbb{U}),$$

where $g(z)$ is given by (10).

Proof. If $f \in \hat{S}^\beta_\alpha$, we get from (9) that

$$\frac{zf'(z)}{f(z)} - 1 = \gamma(1 - \alpha) \left( \frac{zs'(z)}{s(z)} - 1 \right) \quad (z \in \mathbb{U}).$$

Since $g(z)$ is a starlike function, it is obvious that

$$\gamma(1 - \alpha) \left( \frac{zs'(z)}{s(z)} - 1 \right) < \gamma(-\alpha) \left( \frac{1 + z}{1 - z} - 1 \right) = \frac{2\gamma(-\alpha)}{1 - z} \quad (z \in \mathbb{U}).$$

Hence,

$$\frac{zf'(z)}{f(z)} - 1 < \frac{2\gamma(1 - \alpha)}{1 - z} = \frac{zg'(z)}{g(z)} - 1 \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 1.

Combining Theorem 1 and (11), we readily obtain

Corollary. Let $f \in \hat{S}^\beta_\alpha$ and let $g(z)$ is defined by (10). If
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\[ F(z) = J[f](z) \text{ and } G(z) = J[g](z), \]

then

\[ 1 + \frac{zF''(z)}{F'(z)} < 1 + \frac{zG''(z)}{G'(z)} \quad (z \in U). \] (16)

**Lemma 1.** (see [8]). Let \( h \) be convex in \( U \) with \( h(0) = 0 \). If an analytic function \( p \) in \( U \) with \( p(0) = 1 \) satisfies the following subordination condition:

\[ \frac{zp'(z)}{p(z)} < h(z) \quad (z \in U), \]

then

\[ p(z) < q(z) = \exp\left( \int_0^z \frac{h(t)}{t} \, dt \right) \quad (z \in U) \] (17)

and \( q \) is the best dominate.

Making use of Lemma 1, we now prove Theorem 2 below.

**Theorem 2.** Let \( \alpha \in [0, 1), \beta \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) and suppose that

\[ \gamma = e^{-i\beta} \cos \beta. \] If \( f \in S_\alpha^\beta \), then

\[ \frac{f(z)}{z} < \frac{g(z)}{z} \quad (z \in U). \] (18)

where \( g(z) \) is given by (10).

**Proof.** For functions \( f \) and \( g \), we set

\[ F(z) = J[f](z) \text{ and } G(z) = J[g](z). \]

A simple calculation yields

\[ 1 + \frac{zG''(z)}{G'(z)} = \frac{zg'(z)}{g(z)} = 1 + \frac{2\gamma(1 - \alpha)z}{1 - z} \quad (z \in U), \] (19)

where

\[ G(z) = \frac{1 - (1 - z)^{1 - 2\gamma(1 - \alpha)}}{1 - 2\gamma(1 - \alpha)} = _2F_1(2\gamma(1 - \alpha), 1; 2; z) \quad (z \in U). \]
Also, if we put
\[ h(z) = \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \]
then
\[ 1 + \frac{zh''(z)}{h(z)} = \frac{1 + z}{1 - z} \quad (z \in \mathbb{U}). \]

Since
\[ \Re \left( \frac{1 + z}{1 - z} \right) > 0 \quad (z \in \mathbb{U}), \]
h(z) is convex in U. Next, we set
\[ P(z) = F'(z) \quad (z \in \mathbb{U}). \]

From the above Corollary, it is obvious that
\begin{equation}
\frac{zP'(z)}{P(z)} = \frac{zF''(z)}{F'(z)} < \frac{zG''(z)}{G'(z)} = h(z) \quad (z \in \mathbb{U}). \tag{20}
\end{equation}

Hence, by means of the above Lemma 1, we obtain
\begin{equation}
P(z) < \exp \left( \int_0^z \frac{h(t)}{t} \, dt \right) = G'(z) \quad (z \in \mathbb{U}). \tag{21}
\end{equation}

This evidently completes the proof of Theorem 2.

Remark. Theorems 1, 2 generalize the corresponding results of [6], when \( \alpha = 0 \), Theorems 1, 2 were obtained by Y. C. Kim and H. M. Srivastava [6].

References


