HADWIGER’S CONJECTURE FOR GRAPHS WITH INFINITE CHROMATIC NUMBER

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Abstract

We construct a connected graph $H$ such that
(i) $\chi(H) = \omega$;
(ii) $K_\omega$, the complete graph on $\omega$ points, is not a minor of $H$.

Therefore Hadwiger's Conjecture does not hold for graphs with infinite coloring number.

1. Introduction

In this note we are only concerned with simple undirected graphs $G = (V, E)$ where $V$ is a set and $E \subseteq \mathcal{P}_2(V)$ where
\[ \mathcal{P}_2(V) = \{\{x, y\}: x, y \in V \text{ and } x \neq y\}. \]

We also require that $V \cap E = \emptyset$ to avoid notational ambiguities. We denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal $\alpha$ we denote the complete graph on $\alpha$ points by $K_\alpha$.

For any graph $G$, disjoint subsets $S, T \subseteq V(G)$ are said to be connected to each other if there are $s \in S, t \in T$ with $\{s, t\} \in E(G)$. Note

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that $K_\alpha$ is a minor of a graph $G$ if and only if there is a collection
\{ $S_\beta : \beta \in \alpha$ \} of nonempty, connected and pairwise disjoint subsets of
$V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets $S_\beta$ and $S_\gamma$ are
connected to each other. We will need the following observation later on:

**Fact 1.1.** For any graph $G$, finite or infinite, the following are equivalent:

(i) $G$ is connected;

(ii) If $S, T \subseteq V(G)$ are nonempty and disjoint such that $S \cup T = V(G)$
then $S, T$ are connected to each other.

### 2. The Construction

In [1], Hadwiger formulated his well-known and deep conjecture,
linking the chromatic number $\chi(G)$ of a graph $G$ with clique minors. He
conjectured that if $\chi(G) = n \in \mathbb{N}$ then $K_n$ is a minor of $G$. In the
following we present a connected graph $H$ with chromatic number $\omega$ such
that $K_\omega$ is not a minor of $H$. Let $\mathbb{N}$ be the set of positive integers. For
any $n \in \mathbb{N}$ we let

\[ C_n = \{1, \ldots, n\} \times \{n\} \]

and set $V(H) = \bigcup_{n \in \mathbb{N}} C_n$. As for the edge set of $H$, we define

\[ E(H) = \{((1, n), (1, n + 1)): n \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} P_2(C_n). \]

**Proposition 2.1.** $H$ is connected.

**Proof.** Using fact 1.1 we assume we are given non-empty disjoint
subsets $S, T$ of $V(H)$ such that $S \cup T = V(H)$ and we want to show that
$S, T$ are connected to each other. We may assume that $C_1 = \{(1, 1)\} \subseteq S$.
Let $n_0 = \min\{n \in \mathbb{N} : C_n \cap T \neq \emptyset\}$. Note that $n_0 > 1$. We distinguish
two cases: If $(1, n_0) \in T$ then by minimality of $n_0$ we have
$(1, n_0 - 1) \in S$ and $\{(1, n_0 - 1), (1, n_0)\} \in E(H)$ so $S, T$ are connected to
each other. If \((1, n_0) \not\in T\) then \((1, n_0) \in S\), so pick \((b, n_0) \in C_{n_0}\) with \((b, n_0) \in T\), so since \(\mathcal{P}_2(C_{n_0}) \subseteq E(H)\), the sets \(S, T\) are again connected to each other.

**Proposition 2.2.** \(\chi(H) = \omega\).

**Proof.** Since we have \(\text{card } (V(H)) = \omega\) we get \(\chi(H) \leq \omega\). Moreover, each \(C_n\) is a complete subgraph of \(H\), so \(H\) cannot be colored with finitely many colors.

For the remainder of this note, we assume that \(\{S_n : n \in \omega\}\) is a collection of nonempty, connected, pairwise disjoint subsets of \(H\) such that for \(m \neq n\) the sets \(S_n, S_m\) are connected to each other. Our goal is to show that such a collection cannot exist.

First, we need a simple observation on what a connected subset of \(H\) looks like. If \(S \subseteq V(H)\) we define \(I(S) = \{n \in \mathbb{N} : C_n \cap S \neq \emptyset\}\).

**Lemma 2.3.** Suppose \(S \subseteq V(H)\) is connected and \(m < n \in I(S)\). Then for all \(x \in \mathbb{N}\) with \(m \leq x \leq n\) we have \((1, x) \in S\).

**Proof.** If \((1, m) \not\in S\) then \(T = S \cap C_m\) and \(S \setminus T\) are disjoint, nonempty and not connected to each other. By Fact 1.1, \(S\) is not connected, contradicting our assumption. A similar argument shows that \((1, n) \in S\). Suppose there is \(x\) with \(m < x < n\) and \((1, x) \not\in S\). Then set \(T = \{(i, j) \in S : j < x\}\). Again, \(T\) and \(S \setminus T\) are nonempty and not connected to each other, so \(S\) is not connected, contradicting our assumption.

If \(\{S_n : n \in \omega\}\) is a collection of subsets of \(V(H)\) as described above, then for every \(k \in \mathbb{N}\) the set of neighbors of \(S_k\), which is denoted by \(N(S_k)\), must be infinite, as the next lemma shows, this implies that \(I(S_k)\) must be infinite for all \(k \in \mathbb{N}\).

**Lemma 2.4.** If \(S \subseteq V(H)\) is such that \(I(S)\) is finite, then \(N(S)\) is finite.
**Proof.** Let \( m = \max(I_S) \). Then \( N(S) \subseteq \bigcup_{i=1}^{m+1} C_i \), which is a finite set.

Now we go back to our assumption that \( \{S_n : n \in \omega\} \) is a collection of nonempty, connected, pairwise disjoint subsets of \( H \) such that for \( m \neq n \) the sets \( S_m, S_n \) are connected to each other. We consider just two of these sets, say \( S_0, S_1 \). Because of lemma 2.4, the sets \( I(S_0) \) and \( I(S_1) \) are infinite. For \( k = 0, 1 \) let \( \mu_k = \min(I(S_k)) \). We may assume that \( \mu_0 \leq \mu_1 \). Since \( I(S_0) \) is infinite, there is \( n \in I(S_0) \) with \( n \geq \mu_1 \). So lemma 2.3 implies that \( (1, \mu_1) \in S_0 \cap S_1 \), contradicting the assumption that the \( S_k \) are pairwise disjoint. So we established:

**Proposition 2.5.** The complete graph \( K_{\omega} \) is not a minor of \( H \).

**References**