DEGREE EQUITABLE CONNECTED DOMINATION IN GRAPHS

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Abstract
A connected dominating set $D$ is to be an equitable connected dominating set if for every vertex $u \in V - D$ there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|\deg(v) - \deg(u)| \leq 1$. The minimum cardinality of such a connected dominating set is denoted by $\gamma^e_c$ and is called the equitable connected domination number. In this paper, we obtain some bounds for $\gamma^e_c$. Also Nordhaus-Gaddum type results are obtained.

1. Introduction

By a graph, we mean a simple and connected. Any undefined terms in this paper may be found in [2].

A set $D$ of vertices in a graph $G$ is a dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of minimal dominating set of $G$.

A dominating set $D$ is said to be a connected dominating set if $\langle D \rangle$ is connected. The connected domination number $\gamma_c$ of $G$ is the minimum cardinality of a minimal connected dominating set of $G$ [4].
A subset \( D \) of \( V \) is called an equitable dominating set if for every \( v \in V - D \) there exist a vertex \( u \in D \) such that \( uv \in E(G) \) and \( |\deg(u) - \deg(v)| \leq 1 \), where \( \deg(u) \) and \( \deg(v) \) denotes the degree of a vertex \( u \) and \( v \) respectively. The minimum cardinality of such a dominating set is denoted by \( \gamma^e \) and is called the equitable domination number [5].

Analogously, we define degree equitable connected domination as follows:

A connected dominating set \( D \) is to be an equitable connected dominating set if for every vertex \( v \in V - D \) there exists a vertex \( u \in D \) such that \( uv \in E(G) \) and \( |\deg(v) - \deg(u)| \leq 1 \). The minimum cardinality of such a connected dominating set is denoted by \( \gamma^e_c \) and is called the equitable connected domination number.

For illustration, we consider the graph \( G \) as shown in Figure 1.

\[
\begin{align*}
G: & \\
& v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7
\end{align*}
\]

\[Fig. \ 1\]

In Figure 1, the minimal connected dominating set is \( D = \{v_2, v_3, v_8, v_6\} \) and minimal equitable dominating set is \( D' = \{v_1, v_2, v_3, v_8, v_6, v_5\} \). Therefore, \( \gamma_c(G) = |D| = 4 \) and \( \gamma^e_c(G) = |D'| = 6 \).

Hence \( \gamma^e_c(G) > \gamma_c(G) \).

**2. Existence of \( \gamma^e_c \)-Sets**

Obviously, by the definition, \( \gamma^e_c \)-set exist if for every vertex
There exist a vertex $u \in D$ such that $uv \in E(G)$ and $\left| \deg(u) - \deg(v) \right| \leq 1$.

If $\left| \deg(u) - \deg(v) \right| \geq 2$, then $\gamma^e_c$ set does not exist.

There are certain classes of graph such that $\gamma^e_c$ set does not exist.

**Examples:**

(i) $K_{m, n}; \left| m - n \right| \geq 2$.

(ii) $C_p \circ K_1$.

(iii) Tree $T$, such that all the pendant vertices are adjacent to a support vertices of degree at least three.

(iv) $G = H \circ K_1$, where $H$ is any connected graph with $\delta(G) \geq 2$.

3. Results

First, we find the degree equitable connected domination number of some standard graphs.

**Proposition 3.1.**

(i) $\gamma^e_c(K_p) = 1; p \geq 2$

(ii) $\gamma^e_c(P_p) = p - 2p \geq 2$

(iii) $\gamma^e_c(C_p) = p - 2; p \geq 3$

(iv) $\gamma^e_c(W_p) = \begin{cases} 1 & \text{if } p = 4, 5 \\ 3 & \text{if } p = 6 \\ P - 4 & \text{if } p \geq 7 \end{cases}$

(v) $\gamma^e_c(K_{m, n}) = \begin{cases} 2 & \text{if } \left| m - n \right| \leq 1 \\ \text{does not exist} & \text{otherwise} \end{cases}$

**Proof.** (i) For the complete graph $K_p; p \geq 2$, every single vertex forms an equitable connected dominating set. Hence $\gamma^e_c(K_p) = 1; p \geq 2$.

(ii) Since the degree of any vertex in $P_p$ is 2 except the initial and
terminal vertices. Any connected dominating set in $P_p$ is clearly an equitable connected dominating set. Hence $\gamma_c^e(P_p) = \gamma_c(P_p)$. But $\gamma_c(P_p) = p - 2$. Hence $\gamma_c^e(P_p) = p - 2$.

(iii) Proof is similar as in the above case.

(iv) Let $W_p$ be a wheel with $p - 1$ vertices on the cycle and a single vertex at the center. Let $V(W_p) = \{u, v_1, v_2, ..., v_{p-1}\}$, where $u$ is the center and $v_i (1 \leq i \leq p - 1)$ is on the cycle. $\deg(v_i) = 3$ for all $i \leq i \leq p - 1$ and $\deg(u) = p - 1$. Clearly $p \geq 4$. We consider the following cases:

Case 1. $p \leq 5$

Then $\deg(u) = p - 1 \leq 4$. Since $u$ is adjacent with $v_i$ for all $i, 1 \leq i \leq 4$, $\{u\}$ is a connected dominating set of $W_p$. Also $\deg(u)$ is 3 if $p = 4$ and 4 if $p = 5$. In both the cases, $\deg(v_i) = 3; 1 \leq i \leq p - 1$. Therefore, $|\deg(u) - \deg(v_i)| \leq 1$ for all $i, 1 \leq i \leq p - 1$. Therefore, $\{u\}$ is is an equitable connected dominating set of $W_p$.


In this case $\deg(u) \geq 5$, while $\deg(v_i) = 3$ for all $i, 1 \leq i \leq p - 1$. Then $\{u\}$ is a connected dominating set but it is not equitable. Let $D = \{u, v_1, v_3\}$ be a connected dominating set such that $v_1$ and $v_2$ are independent, which is equitable. Hence $\gamma_c^e = |D| = 3$.

Case 3. $p \geq 7$.

In this case $\deg(u) \geq 6$ while $\deg(v_i) = 3$ for all $i, 1 \leq i \leq p - 1$. Then $D = \{u, v_1, v_2, ..., v_k, v'_1, v'_2, ..., v'_l\}$ form a equitable connected dominating set such that each $v_i, 1 \leq i \leq k$ are adjacent and $v'_i, 1 \leq i \leq l$ are adjacent but $v_k$ and $v_l$ are independent and each vertex in $v_k$ and $v_l$ are adjacent to $u$, such that $\langle D \rangle$ is connected and $|D| = p - 4$. Hence $D$ is a equitable connected dominating set. Hence $\gamma_c^e(W_p) = |D| = p - 4$. 
Let \( K_{m,n} \) be complete bipartite graph with \( m \) vertices in one partition say \( V_1 \) and \( n \) vertices in another partition say \( V_2 \). Then,

\[
\deg(u) = \begin{cases} n, & u \in V_1 \\ m, & u \in V_2 \end{cases}
\]

If \(|m - n| \leq 1\), then any vertex say \( u \) from \( V_1 \) and any vertex \( v \) from \( V_2 \) constitute a connected dominating set, which is equitable. Therefore \( \gamma^e_c(K_{m,n}) \leq 2 \) if \(|m - n| \leq 1\). Since \( m, n \geq 2 \), \( K_{m,n} \) has no vertex which is adjacent to every other vertex. Therefore, \( \gamma^e_c \geq 2 \). Also, \( \gamma^e_c(K_{m,n}) = 2 \) if \(|m - n| \leq 1\).

If \(|m - n| \geq 2\), then obviously the equitable connected dominating set does not exist.

Next, we obtain the upper bound for \( \gamma^e_c(G) \).

**Theorem 3.1.** For any graph \( G \), \( \gamma^e_c(G) \leq p - 1 \).

Further, equality holds if \( G = T \), such that there exists a unique pendant vertex which is adjacent to a support vertex of degree two.

**Proof.** Let \( G \) be a connected graph. Then by Proposition 3.1, the upper bound is obvious.

For equality, let \( T \) be a tree of order at least two. Let \( V(T) = \{v_1, v_2, ..., v_p\} \). If there exist a unique pendant \( v_i \) which is adjacent to a support vertex \( v_j \) of degree two. Then \( D = \{v_1, v_2, ..., v_j\} \) be a connected dominating set. Such that \( V(T) - D = \{v_i\} \). Clearly \( D \) is equitable. Hence \( \gamma^e_c(T) = |D| = p - 1 \).

**Theorem 3.2.** For any graph \( G \), \( \gamma^e_c(G) \leq 2q - p + 1 \)

**Proof.** Let \( G \) be any \((p, q)\) graph. Then by Theorem 3.1

\[
\gamma^e_c \leq p - 1 \\
\leq 2(p - 1) - p + 1 \\
\leq 2q - p + 1
\]
Further, equality holds if $G$ is a tree and there exist a unique pendant vertex which is adjacent to a support vertex of degree two.

**Corollary 1.** If $G$ is a path then, $\gamma_e^c(G) = 2q - p$.

**Theorem 3.3.** Let $H$ be a spanning subgraph of $G$, then $\gamma_e^c(G) \leq \gamma_e^c(H)$.

**Proof.** Let $G$ be any $(p, q)$ graph. By Theorem 3.1, $\gamma_e^c(G) \leq p - 1$. Let $H$ be any spanning subgraph of $G$. If $H$ is a spanning tree and there exist a unique pendant vertex which is adjacent to a support vertex of degree two then $\gamma_e^c(H) = p - 1$. Hence $\gamma_e^c(G) \leq \gamma_e^c(H)$.

4. Comparison of $\gamma_e^c$ with other Domination Parameters

**Observation.** For any connected graph $G$

$$\gamma(G) \leq \gamma_e^c(G) \text{ and } \gamma_c(G) \leq \gamma_e^c(G).$$

**Theorem 4.1.** If $G$ is a regular graph, then $\gamma_e^c(G) = \gamma_c(G)$

**Proof.** Suppose $G$ is a regular graph. Then every vertex has the same degree say $k$. Let $D$ be a minimum connected dominating set of $G$. Then $|D| = \gamma_c(G)$. Let $u \in V - D$. Then as $D$ is a connected dominating set, there exist a vertex $v \in D$ and $uv \in E(G)$. Also $\deg(u) = \deg(v) = k$. Therefore, $|\deg(u) - \deg(v)| = 0 < 1$. Hence $D$ is an equitable connected dominating set of $G$, so that $\gamma_e^c(G) \leq |D| = \gamma_c(G)$. But $\gamma_c(G) \leq \gamma_e^c(G)$. Therefore, $\gamma_e^c(G) = \gamma_c(G)$.

Next, we characterize the graphs with $\gamma_e^c(G) = \gamma_e^c$.

**Lemma 1.** Let $G$ be any connected graph and let $D$ be minimum equitable dominating set of $G$, then $\Delta(D) < \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of a vertex in $G$.

**Proof.** Let $D$ be a minimum equitable dominating set of $G$. Suppose $\Delta(D) = \Delta(G)$. Let $v \in D$ be such that $\deg(D)(v) = \Delta(G)$. Then $D - \{v\}$ is a dominating set of cardinality $\gamma_e^c(G) - 1$, a contradiction.
Theorem 4.2. Let $G$ be a connected cubic graph of order $p$ and let $D$ be a minimum equitable dominating set of $G$. If $\gamma^e(G) = \gamma_c^e(G)$, then

(i) $\langle D \rangle$ is a path

(ii) Every vertex in $V - D$ is adjacent to exactly one vertex in $D$

(iii) $p = 2(\gamma^e(G) + 1)$

(iv) $\gamma^e_c(G) \leq 3$

Proof. (i) Let $G$ be a connected 3-regular graph. Let $D$ be a minimum equitable connected dominating set of $G$, such that for every $u \in V - D$ there exist a vertex $v \in D$ and $uv \in E(G)$. Therefore $|\deg(u) - \deg(v)| = 0$. So that $\gamma_c^e = |D| = \gamma^e(G)$. But by Lemma 1, $\Delta(\langle D \rangle) < 3$ and hence $\langle D \rangle$ is either a path or a cycle. Suppose $\langle D \rangle$ is a cycle $C = u_1, u_2, ..., u_n, u_1$. Let $v_i$ be a unique vertex in $V - D$ such that $v_i$ is adjacent to $u_j$ and $|\deg (u_i) - \deg (v_i)| = 0$. Clearly $v_iu_j \notin E(G)$ for $i \neq j$ and hence $v_i$ is adjacent to two vertices in $V - D$, say $v_j$ and $v_k$. Now $(D - \{v_jv_k\} \cup \{v_i\})$ is an equitable dominating set of cardinality $\gamma^e(G) - 1$, which is a contradiction. Hence $\langle D \rangle$ is a path.

(ii) Suppose every vertex in $V - D$ is not adjacent to exactly one vertex in $D$. Then there exist a vertex $v \in V - D$ is adjacent to two vertices $u_i$ and $u_j$ in $\langle D \rangle$ such that $\deg(u_i) = \deg(u_j) \geq 2$. Then $(S - \{u_i, u_j\}) \cup \{v\}$ is an equitable dominating set of cardinality $\gamma^e(G) - 1$. If $v$ is adjacent to a vertex $v_i$ of degree at least two and a pendant vertex in $\langle D \rangle$, say $u_i$. Then $D - \{u_i\}$ is an equitable dominating set of cardinality $\gamma^e(G) - 1$. If $v$ is adjacent to the pendant vertices $u_1$ and $u_n$ in $\langle D \rangle$ then $D - \{u_n\}$ is an equitable dominating set of cardinality $\gamma^e(G) - 1$. Thus $v$ is adjacent to exactly one vertex in $D$.

(iii) It follows from (i) and (ii) that each pendant vertex in $\langle D \rangle$ is adjacent to two distinct vertices in $V - D$. So that
\[ |V - D| = |S| + 2 = \gamma_e^c(G). \] Hence \( p = |V| = 2(\gamma_e^c(G) + 1). \)

(iv) Suppose \( \gamma_e^c(G) = m \geq 4. \)

Let \( D = \{u_1, u_2, ..., u_m\} \) be a minimum connected dominating set with \( \langle D \rangle = u_1, u_2, ..., u_m. \) Let \( V - D = \{w_1, w_2, ..., v_1, v_2, ..., v_m\}, \) where \( u_iv_i \in E(G) \) and \( |\deg(u_i) - \deg(v_i)| = 0 < 1, \) for all \( i = 1, 2, ..., m, \) where \( w_1u_1, w_2u_m \in E(G) \) and \( u_iv_j \notin E(G) \) if \( i \neq j. \) Let \( D_1 = \{v_1w_1\}, D_2 = \{v_m, w_2\} \) and \( D_3 = \{v_2, v_3, ..., v_{m-1}\}. \) If two vertices in \( D_3, \) say \( v_i \) and \( v_j \) are adjacent then \( (D - \{u_i, u_j\}) \cup \{v_i\} \) is an equitable dominating set of cardinality \( \gamma_e^c(G) - 1, \) which is a contradiction.

Hence \( D_3 \) is independent. By similar argument we can prove no vertices in \( D_3 \) are adjacent to the same vertex in \( D_1 \cup D_2 \) so that \( |D_3| = 2 \) and \( \gamma_e^c = m = 4. \) Now, since \( G \) is a cubic graph. We have the following cases.

**Case (i)** \( v_2 \) is adjacent to the vertices in \( D_1. \) Since no two vertices in \( D_3 \) are adjacent to the same vertex in \( D_1 \cup D_2, v_3 \) must be adjacent to the two vertices of \( D_2. \) The remaining two edges are \( w_1v_1, w_2v_4 \) or \( w_1v_4, v_1w_2 \) or \( w_1u_2, v_1v_4 \) so that \( \{u_1, u_3, u_4\} \) or \( \{w_1, w_2, u_3\} \) or \( \{w_1, v_4, u_2\} \) is an equitable dominating set of cardinality 3, which is a contradiction.

**Case (ii)** \( v_2 \) is adjacent to two vertices in \( D_2. \) In this case \( \{v_2, u_3, u_1\} \) is an equitable dominating set, which is a contradiction.

**Case (iii)** \( v_2 \) is adjacent to one vertex in \( S_1 \) and one vertex in \( S_2. \) Let \( v_2 \) be adjacent to \( v_1 \) and \( v_4. \) Then \( v_3 \) is adjacent to \( w_1 \) and \( w_2 \) so that \( \{u_1, v_3, v_4\} \) is an equitable dominating set, which is a contradiction.

Thus, \( \gamma_e^c(G) \leq 3. \)
Theorem 4.3. Let $G$ be connected cubic graph of order $p$. Then $\gamma^e(G) = \gamma^e_c$ if and only if $G \equiv K_4$, $\overline{C_6}$, $K_{3,3}$, $G_1$ or $G_2$.

Proof. If $G \equiv K_4$, $K_{3,3}$, then by Theorem 4.1, $\gamma^e(G) = \gamma^e_c(G)$. If $G \equiv \overline{C_6}$, then $\gamma^e(G) = 2 = \gamma^e_c(G)$. When $G \equiv G_1$ or $G_2$, then $\gamma^e(G) = 3 = \gamma^e_c(G)$.

Now, let $G$ be cubic graph of order $p$. Let $\gamma^e_c(G) = \gamma^e(G)$ and $D$ be a minimum equitable dominating set of $G$. If $\gamma^e_c(G) = 1$ then $G = K_4$. If $\gamma^e_c = 2$, then by Theorem 4.2, $p = 6$. Also $\overline{G}$ is regular of degree two. Hence $\overline{G} \equiv C_6$ or $C_3 \cup C_3$ so that $G$ is isomorphic to $\overline{C_6}$ or $K_{3,3}$.

If $\gamma^e_c(G) = 3$, then by Theorem 4.2, $p = 8$. Let $D = \{u_1, u_2, u_3\}$ be a minimum equitable connected dominating set in $G$ and let $\langle D \rangle = u_1u_2u_3$. Let $V - D = \{v_1, v_2, v_3, w_1, w_2\}$ where $u_iv_i$, $u_1w_1$, $u_3w_3 \in E(G)$. Let $D_1 = \{v_1, w_1\}$ and $D_2 = \{v_3, w_2\}$. We consider the following cases.

Case (i) $\langle D_1 \rangle = \langle D_2 \rangle = K_2$.

In this case $v_2$ is adjacent to some vertex in $D_1 \cup D_2$, say $v_1$. Now $\{v_1, u_3\}$ is a dominating set in $G$, which is a contradiction.

Case (ii) $D_1$ and $D_2$ are independent. Since $G$ is cubic, $v_2$ is adjacent to one vertex in $D_1$ and one vertex in $D_2$. Without loss of generality, we assume that $u_2v_1$, $u_2v_3 \in E(G)$. Then $v_1$ must be adjacent to
$w_2$ and $w_1$ must be adjacent to $v_3$ and $w_2$ so that $G$ is isomorphic to $G_1$.

**Case (iii)** $\langle D_1 \rangle = K_2$ and $D_2$ is independent. In this case $v_2$ must be adjacent to $v_3$ and $w_2$. Without loss of generality, let $v_1$ be adjacent to $v_3$ and $w_1$ be adjacent to $w_2$. Then $G$ is isomorphic to $G_2$.

### 5. Nordhaus-Gaddum Type Results

**Theorem 4.4.** Let $G$ and $\overline{G}$ be connected graphs of order at least two. Then

$$\gamma_c^e(G) + \gamma_c^e(\overline{G}) \leq p(p - 3).$$

Further, equality holds if and only if $G = P_4$.

Finally, we conclude this paper by exploring the following two open problems.

### 6. Open Problems

(i) Characterize the graphs in which $\gamma_c(G) = \gamma_c^e(G)$.

(ii) Characterize the graphs in which $\gamma_c^e(G) = \gamma_c(G)$.

### References


