ON THE NUMBER OF EMPTY CONVEX QUADRILATERALS IN A PLANAR 26-POINT SET

YATAO DU, HANYING FENG, XINGFANG FENG and HUIXUAN TAN

Department of Mathematics
Shijiazhuang Mechanical Engineering College
Shijiazhuang, 050003
P. R. China
e-mail: dyt77@eyou.com

Abstract

Let \(P\) be a set of 26 points in general position in the plane. In this paper, we consider the problem about how many disjoint empty convex 4-gons can be constructed in \(P\).

1. Introduction

We say a finite planar point set \(P\) is in general position if no three points of \(P\) are collinear. Let \(P\) be a finite planar point set in general position. A convex polygon determined by a subset of \(P\) is called to be empty if no points of \(P\) lie in the interior of its convex hull. In 2001, K. Hosono and M. Urabe [1] considered the following question: How many disjoint empty convex \(k\)-gons can be constructed in a planar point set for a fixed \(k\)? They mainly studied \(k = 4\) and posed an outstanding question about the number of disjoint empty convex 4-gons in a 26-point set. In this paper, we discuss the related question.

We call a partition of \(P\) a disjoint convex partition if \(P\) is partitioned by subsets \(S_1, S_2, \ldots, S_t\), such that each \(CH(S_i)\) is an \(|S_i|\)-gon and \(CH(S_i) \cap CH(S_j) = \emptyset (i, j = 1, 2, \ldots, t; i \neq j)\) (see [1], [4], [5]).
Let $k$ be a positive integer and $\Pi_k^\pi(P)$ be the number of convex $k$-gons in a disjoint partition $\pi$ of $P$. We denote

$$f_k(P) = \max\{\Pi_k^\pi(P) : \pi \text{ is a disjoint partition of } P\}$$

Let $C(a; b, c)$ denote the convex cone determined by $\{a, b, c\} \subset P$ such that $a$ is the apex and $b$ and $c$ are on the boundary of the cone (see [2], [3]). By $A(a; b, c)$ we denote a point of $P$ in $C(a; b, c)$ such that $C(a; b, A(a; b, c))$ is empty.

In the sequel, if not otherwise stated, $P$ always denotes a set of 26 points in general position. We call a line $l$ a cutting line of $P$ if $l$ separates the plane into two open half planes $H_1$ and $H_2$ such that $H_1 \cup l$ contains a convex quadrilateral determined by a subset of $P$ and that $H_1$ does not contain the remaining 22 points of $P$. Moreover, we use the notation $\overline{ab}$ to refer to the extended straight line associated with the points $a$ and $b$.

2. Main Results

We consider $f_4(P)$ for all possible $P$ with $|P| = 26$ and find $f_4(P)$ for the 26-point planar sets $P$ with $|V(P)| = 3$ and $|V(P)| \geq 13$. We need the following lemmas.

**Lemma 1.** (see [1]) Any 9 point set with no three collinear can be partitioned into 2 disjoint empty convex 4-gons so that 1 point remains.

**Lemma 2.** (see [1]) For any set of $2m + 4$ points in the plane, no three collinear, we can divide the plane into three disjoint convex regions such that one contains a convex quadrilateral and the others contain $m$ points each, where $m$ is a positive integer.

**Lemma 3.** Let $P$ be a set of 26 points in the plane, no three collinear. If there exists a cutting line of $P$, then $f_4(P) = 6$.

**Proof.** Let $m = 9$ in Lemma 2, then it’s obvious that for a set of 22 points we can construct 5 disjoint convex quadrilaterals by Lemma 1. So we can get 6 disjoint convex quadrilaterals by the definition of the cutting line.
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For convenience we use the following notations. Let $V(P)$ denote the vertices of $CH(P)$ and $Q = P \setminus V(P)$. Let $V(P) = \{v_1, v_2, \ldots, v_m\}$, $V(Q) = \{y_1, y_2, \ldots, y_n\}$. We denote by $(\ast)$ the assumption that $P$ has a cutting line.

**Lemma 4.** If $(\ast)$ is not true, then

(i) There exists a point $p_i$ of $I(P)$ for every edge $v_i \overline{v}_{i+1}$ of $CH(P)$ such that $C(v_i; p_i, v_{i+1}) \cup C(v_{i+1}; p_i, v_i)$ is empty. Such point $p_i$ is called the characteristic point to $v_i v_{i+1}$. Characteristic points are different from each other.

(ii) $|V(P)| \leq \frac{26}{2} = 13$.

(iii) The outside of the extended straight line associated with each edge of $CH(Q)$ contains only one point of $V(P)$.

**Proof.** (i) If $\Delta v_i-1v_i v_{i+1}$ is empty, we can separate the empty quadrilateral $v_i-1 v_{i+1} A(v_{i+1}; v_i-1, v_i-2)$ from the remaining 22 points and thus we get a cutting line, a contradiction. So $\Delta v_i-1v_i v_{i+1}$ is not empty. Let $p_i = A(v_{i+1}; v_i, v_{i-1})$, then clearly $C(v_{i+1}; p_i, v_i)$ is empty. $C(v_{i-1}; p_i, v_{i+1})$ is also empty, since otherwise the empty quadrilateral $p_i v_i v_{i+1} A(p_i; v_{i+1}, v_i+2)$ can be separated, which leads to $(\ast)$, again a contradiction.

Next we show that characteristic points are different from each other. If $p_i = p_{i-1}$, then $C(v_{i-1}; p_i, v_i)$ is also empty, and the empty quadrilateral $v_{i-1} v_i A(p_i; v_{i-1}, v_i-2)$ can be separated, thus $(\ast)$ holds, a contradiction.

(ii) Let $t$ be the number of characteristic points, then

$$|V(P)| = t \leq |I(P)| \Rightarrow |V(P)| \leq 26 - |V(P)| \Rightarrow |V(P)| \leq \frac{26}{2} = 13.$$

(iii) If the outside of the extended straight line associated with an edge of $CH(Q)$ contains more than one point of $V(P)$, then $(\ast)$ is true, a contradiction.
**Theorem 1.** If \(| P | = 26, | V(P) | \geq 13\), then \(f_4(P) = 6\).

**Proof.** When \( | V(P) | > 13\), \((*)\) is true by Lemma 4 (ii), and hence \(f_4(P) = 6\) by Lemma 3. When \( | V(P) | = 13\), if \((*)\) holds, the conclusion is reached; if \((*)\) doesn’t hold, let \(t\) be the number of characteristic points, then \(t = | I(P)| = 13\), and 6 disjoint empty convex quadrilaterals can be constructed as shown in Figure 1, and the conclusion is reached.

**Theorem 2.** If \(| P | = 26, | V(P) | = 3\) and \(CH(Q)\) is a \(k\)-gon with \(3 \leq k \leq 5\), then \(f_4(P) = 6\).

**Proof.** Let \(V(P) = \{v_1, v_2, v_3\}\). Three cases need to be considered: \(k = 3, 4, 5\). For each side of \(CH(P)\) there exists a point of \(Q\) which is closest to the side. We call the closest point a near point of the side. If a side has two near points then obviously the straight line through the two near points is a cutting line and the conclusion is reached. If two sides have a common near point, then a cutting line is determined. So in the sequel we assume that each side has a unique near point, and three near points are different from each other.

**Case 1.** \(k = 3\).

Let \(V(Q) = \{y_1, y_2, y_3\}\), where \(y_1, y_2, y_3\) are near points.

If \(C(y_1; y_2, v_2)\) is not empty, let \(q = A(y_1; y_2, v_2)\), then the quadrilateral \(v_1y_1qy_2\) is empty and convex and \(y_1q\) is a cutting line. If \(C(y_1; y_2, v_2)\) is empty, let \(q_1 = A(y_1; y_3, v_2)\), then the quadrilateral \(v_3y_1q_1y_3\) is empty and convex and \(y_1q_1\) is a cutting line.

**Case 2.** \(k = 4\).

Let \(V(Q) = \{y_1, y_2, y_3, y_4\}\), where \(y_2, y_3, y_4\) are near points.
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Suppose $y_1$ is on the left side of $v_1y_3$. If $\Delta y_1y_3y_4$ is empty, then we have an empty convex quadrilateral $v_3y_3y_1y_4$ and $y_3y_1$ is a cutting line. If $\Delta y_1y_3y_4$ is not empty, let $q = A(y_3; y_4, y_1)$, then we have an empty convex quadrilateral $v_3y_3qy_4$ and $y_3q$ is a cutting line.

We reach the conclusion similarly when $y_1$ is on the right side of $v_1y_3$.

**Case 3.** $k = 5$. Let $V(Q) = \{y_1, y_2, y_3, y_4, y_5\}$, in which there are two non-near points, see Figure 2.

**Subcase 3.1.** Two non-near points $y_1, y_3$ are not consecutive, as shown in Figure 2 (a).
Suppose $y_1$ is on the left side of $v_1y_4$. By an argument similar to that in case 2, we obtain the conclusion.

Suppose $y_1$ is on the right side of $v_1y_4$.

(i) $\Delta y_1y_4y_5$ is not empty. Let $q = A(y_4; y_5, y_1)$. If $q$ is on the left side of $v_1y_4$, by an argument similar to that in case 2, we prove this case. If $q$ is on the right side of $v_1y_4$, we consider $C(v_1; y_4, q)$. When $C(v_1; y_4, q)$ is empty, $v_1q$ is a cutting line with $v_3y_5qy_4$ being an empty convex quadrilateral. When $C(v_1; y_4, q)$ is not empty, let $q_1 = A(v_1; y_4, q)$, $v_1q_1$ is a cutting line with $v_3y_5q_2y_4$ being an empty convex quadrilateral.

(ii) $\Delta y_1y_4y_5$ is empty. If $C(v_1; y_4, y_1)$ is empty, then $v_1y_1$ is a cutting line with $v_3y_5y_1y_4$ being an empty convex quadrilateral. If $C(v_1; y_4, y_1)$ is not empty, let $q_2 = A(v_1; y_4, y_1)$, then $v_1q_2$ is a cutting line with $v_3y_5q_2y_4$ being an empty convex quadrilateral.

Subcase 3.2. Two non-near points $y_1, y_2$ are consecutive, as shown in Figure 2 (b).

By an argument similar to that in case 2, we obtain the conclusion.

Remark. Let

$$F_k(n) = \min \{f_k(P); |P| = n\}$$

From [1] we have $F_4(26) \geq 5$, and obviously $F_4(26) \leq 6$, so $F_4(26) = 5$ or 6. It is known that $F_4(27) = 6$. Our results made a progress toward the complete solution of the the conjecture $F_4(26) = 6$ (see [1]).

References


