SOME NEW PARANORMED SEQUENCE SPACES OF NONABSOLUTE TYPE AND MATRIX TRANSFORMATIONS

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Abstract

In the present paper, we introduce the sequence spaces \( a_\alpha^c(u, \Delta) \) and \( a_\gamma^c(u, \Delta) \). We show their completeness property, study their linear isomorphism property and compute their \( \alpha, \beta \) - and \( \gamma \)-duals. Furthermore, we construct the basis of \( a_\alpha^c(u, \Delta) \) and \( a_\gamma^c(u, \Delta) \). In our last section we characterize some matrix class.

1. Preliminaries, Background and Notation

Let \( \omega \) denote the space of all sequences (real or complex). The family under pointwise addition and scalar multiplication forms a linear (vector) space over real of complex numbers. Any subspace of \( \omega \) is called the sequence space. So the sequence space is the set of scalar sequences (real of complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let \( l_\infty, c \) and \( c_0 \), respectively, denotes the space of all bounded sequences, the space of convergent sequences and the space of convergent sequences and the

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sequences converging to zero. Also, by $cs$, $l_1$ and $l(p)$ we denote the spaces of all convergent, absolutely and $p$-absolutely convergent series, respectively. A linear Topological space $X$ over the field of real numbers $R$ is said to be a paranormed space if there is a subadditive function $h : X \rightarrow R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \rightarrow 0$ and $h(x_n - x) \rightarrow 0$ imply $h(\alpha_n x_n - \alpha x) \rightarrow 0$ for all $\alpha's$ in $R$ and $x's$ in $X$, where $\theta$ is a zero vector in the linear space $X$. For the sequence space $X$ and $Y$, define the set

$$S(X : Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}. \quad (1)$$

With the notation of (1), the $\alpha$-, $\beta$- and $\gamma$-duals of a sequence space $X$, which are respectively denoted by $X^\alpha$ and $X^\beta$ and are defined by

$$X^\alpha = S(X : l_1) \quad \text{and} \quad X^\beta = S(X : cs).$$

If a sequence space $X$ paranormed by $h$ contains a sequence $(b_n)$ with the property that for every $x \in X$ there is a unique sequence of scalars $(\alpha_n)$ such that

$$\lim_{n} h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0$$

then $(b_n)$ is called a Schauder basis (or briefly basis) for $X$. The series $\sum \alpha_k b_k$ which has the sum $x$ is then called the expansion of $x$ with respect to $(b_n)$ and written as $x = \sum \alpha_k b_k$.

Let $X$, $Y$ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, the matrix $A$ defines the $A$-transformation from $X$ into $Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = ((Ax)_n)$, the $A$-transform of $x$ exists and is in $Y$; where $(Ax)_n = \sum_{k} a_{nk} x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $A \in (X : Y)$ we mean the characterizations of matrices from $X$ to $Y$ i.e., $A : X \rightarrow Y$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$
SOME NEW PARANORMED SEQUENCE SPACES OF...

converges to $l$ which is called as the $A$-limit of $x$. For a sequence space $X$, the matrix domain $X_A$ of an infinite matrix $A$ is defined as

$$X_A = \{x = (x_k): x = (x_k) \in \omega\}. \quad (2)$$

We shall denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. Also, we shall write $e^{(k)}$ for the sequence whose only non-zero term is 1 at the $k$th place for each $k \in \mathbb{N}$. The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors, (see, [1, 6, 11, 12, 16, 19]). They introduced the sequence spaces $(l_\infty)_N$ and $c_N$ (see, [19]), $(l_p)_{c_1} = X_p$ and $(l_\infty)_{c_1} = X_\infty$ (see, [15]), $\mu_G = Z(u, u, \mu)$ (see, [12]), $(l_\infty)_{R^f} = r^f_\omega$, $(c)_{R^f} = r^f_c$ and $(c_0)_{R^f} = r^f_0$ (see, [11]), $(l_p)_{R^f} = r^f_p$ (see, [1]), $(c_0)_{E^r} = e^r_0$ and $(c)_{E^r} = e^r_c$ (see, [2]), $(l_p)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$ (see, [3]), $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ (see, [4]), $[c_0(u, p)]_{A^r} = a^r_0(u, p)$ and $[c(u, p)]_{A^r} = a^r_0(u, p)$ (see, [5], $(l_p)_{A^r} = a^r_p$ and $(l_\infty)_{A^r} = a^r_\infty$ (see, [6]), and $(c_0)_{c_1} = c_{01}$, $c_{C_1} = c$ (see, [16], $c_{C_1}^\Lambda(\Delta) = (c_0^\Lambda)_\Delta$ and Neyaz and Hamid $r^{\Delta}(u, p) = \|l(p)\|_{R^q}$ [17]; where $N_q$, $C_1R^f$ and $E^r$ denotes the Nörland, Cesàro, Riesz and Eular means, respectively, $A^r$ and $C$ are respectively defined in [12], $\mu = \{c_0, c, l_p\}$ and $1 \leq p < \infty$. In the present paper, following (see, [1, 6, 11, 12, 16, 19]), we introduce the sequence spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$. Furthermore, we determine the $\alpha$, $\beta$, and $\gamma$-duals of these spaces and construct basis for these spaces. In the last section of the paper we characterize some matrix classes concerning these spaces.

Kizmaz [9] defined the difference sequence spaces $Z(\Delta)$ as follows

$$Z(\Delta) = \{x = (x_k) \in \omega: (\Delta x_k) \in Z\}$$

where $Z \in \{l_\infty, c, c_0\}$ and $\Delta x_k = x_k - x_{k+1}$.

Basar and Altay [7] has studied the sequence space as
With the notation of (2), the space $bv_p$ can be redefined as

$$bv_p = (l_p)_\Delta, 1 \leq p < \infty$$

where, $\Delta$ denotes the matrix $\Delta = (\Delta_{nk})$ defined as

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \leq k \leq n, \\ 0, & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

2. The Sequence Spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$ of Non-absolute Type

In this section, we define the spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$, and prove that these spaces $BK$-space and show these are linearly isomorphic to the spaces $c_0$ and $c$, respectively. We also compute $\alpha$ and $\beta$-duals of these spaces and finally we give the basis for the spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$.

Following Basar and Altay [7], Altay, Basar and mursaleen [3], Choudhary and Mishra [8], Mursaleen and Noman [13], Mursaleen, Basar and Altay [14], Neyaz and Hamid [17], we define the sequence spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$ as follows

$$a_0^r(u, \Delta) = \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n+1} \sum_{k=0}^{n} u_k \Delta x_k \right\} = 0$$

and

$$a_c^r(u, \Delta) = \left\{ x = (x_k) \in \omega : \lim_{n} \frac{1}{n+1} \sum_{k=0}^{n} u_k \Delta x_k \text{ exists} \right\},$$

where, $\Delta x_k = x_k - x_{k-1}$. Incase, $(u_k)_e = e = (1, 1, ...)$ and $\Delta x_k = x_k$ (fixed) for all $k \in \mathbb{N}$, the spaces $a_0^r(u, \Delta)$ and $a_c^r(u, \Delta)$ reduces to the spaces $a_0^r$ and $a_c^r$, introduced by Aydin and Basar [4].
With the notation of (2) that

\[ a_0^r(u, \Delta) = \{a_0^r\}_{\Delta} \text{ and } a_c^r(u, \Delta) = \{a_c^r\}_{\Delta}. \]

Define the sequence \( y = \{y_n(r)\} \), which will be used, by the \( A^r \)-transform of a sequence \( x = (x_k) \), i.e.,

\[ y_n(r) = \sum_{k=0}^{n-1} \frac{1 - r^k}{1 + n} u_k x_k + \frac{1 - r^n}{1 + n} u_n x_n. \] \hspace{1cm} (3)

Now, we begin with the following theorem which is essential in the text.

**Theorem 2.1.** The spaces \( a_0^r(u, \Delta) \) and \( a_c^r(u, \Delta) \) are BK-spaces with the norm

\[ \|x\|_{a_0^r(u, \Delta)} = \|x\|_{a_c^r(u, \Delta)} = \|y\|_{\ell_\infty}. \]

**Proof.** The proof is an easy exercise so is left for the reader.

Note that one can easily see the absolute property does not hold on the spaces \( a_0^r(u, \Delta) \) and \( a_c^r(u, \Delta) \), that is \( \|x\|_{\Delta} \neq \|x\|_{\Delta} \) for at least one sequence in the spaces \( a_0^r(u, \Delta) \) and \( a_c^r(u, \Delta) \) and this says that \( a_0^r(u, \Delta) \) and \( a_c^r(u, \Delta) \) are sequence spaces of non-absolute type.

**Theorem 2.2.** The sequence spaces \( a_0^r(u, \Delta) \) and \( a_c^r(u, \Delta) \) are linearly isomorphic to the spaces \( c_0 \) and \( c \), respectively.

**Proof.** To prove the theorem, we should show the existence of a linear bijection between the spaces \( a_c^r(u, \Delta) \) and \( c \). With the notation of (3), define the transformation \( T \) from arc \( a_c^r(u, \Delta) \) to \( c \) by \( x \rightarrow y = Tx \). The linearity of \( T \) is trivial. Further, it is obvious that \( x = \theta \) whenever \( Tx = \theta \) and hence \( T \) is injective.

Let \( y \in c \) and define the sequence \( x = (x_k) \) by

\[ x_k = \sum_{k=0}^{n} \sum_{i=k-1}^{k} (-1)^{k-i} \frac{1 + i}{(1 + r^k)u_k} y_i \text{ for } k \in N. \]
Then, as

\[(\Delta x)_n = \sum_{i=n+1}^{k} (-1)^{k-i} \frac{1+i}{(1+r^n)u_n} y_i \quad k \in \mathbb{N},\]

we have

\[
\lim_{n} \frac{1}{1+n} \sum_{k=0}^{n} (1+r^k)u_k \sum_{i=k-1}^{k} (-1)^{k-i} \frac{1+i}{(1+r^k)u_k} y_i = \lim_{n} y_n \text{ exists.}
\]

Thus, we have that \(x \in a_c^r(u, \Delta)\). Moreover, we have

\[
\| x \|_{a_c^r(u, \Delta)} = \sup_{n} \left| \frac{1}{n+1} \sum_{k=0}^{n} (1+r^k)u_k (x_k - x_{k-1}) \right| = \sup_{n} \left| \frac{1}{n+1} \sum_{k=0}^{n} [(k+1)y_k - k y_{k-1}] \right| = \| y \|_\infty.
\]

Consequently, \(T\) is surjective and is norm preserving. Hence, \(T\) is a linear bijection and this says us that the spaces \(a_c^r(u, \Delta)\) and \(c\) are linearly isomorphic. Similarly, if we replace the spaces \(a_c^r(u, \Delta)\) and \(c\) respectively by \(a_0^r(u, \Delta)\) and \(c_0\), we shall obtain \(a_0^r(u, \Delta)\) and \(c_0\) are linearly isomorphic, hence the proof.

3. The \(\alpha-, \beta-\) and \(\gamma-\) Duals and Basis of the Spaces \(a_0^r(u, \Delta)\) and \(a_c^r(u, \Delta)\)

In this section, we state and prove the theorems determining the \(\alpha-, \beta-\) and \(\gamma-\) duals and construct the basis of the spaces \(a_0^r(u, \Delta)\) and \(a_c^r(u, \Delta)\).

First we first state some lemmas, due to Stieglitz and Tietz (see, [18]), which are needed in proving our theorems.

**Lemma 3.1.** Let \(A \in (c_0 : l_1)\) if and only if

\[
\sup_{K \in P} \sum_{n} \left| \sum_{k \in K} a_{nk} \right| < \infty.
\]
Lemma 3.2. A ∈ (c₀ : c) if and only if
\[ \lim_{n} a_{nk} = \alpha_k, \quad k \in \mathbb{N}, \] \[ (4) \]
\[ \sup_n \sum_k |a_{nk}| < \infty. \] \[ (5) \]

Lemma 3.3. A ∈ (c : c) if and only if (4) and (5) hold along with
\[ \lim_n \sum_k a_{nk} = \alpha, \] \[ (6) \]
also holds.

Lemma 3.4. A ∈ (c : l_\infty) = (c : l_\infty) if and only if (5) holds.

Theorem 3.5. Define the sets \( D_1, D_2, D_3 \) and \( D_4 \) as follows
\[ D_1 = \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} c_{nk}^r \right| < \infty \right\}, \]
where \( C^r = (c_{nk}^r) \) is defined viz., the sequence \( a = (a_n) \) by
\[ c_{nk} = \begin{cases} 
(1 + k) \left( \frac{1}{(1 + r^k)u_k} - \frac{1}{(1 + r^{k+1})u_{k+1}} \right) a_n, & \text{if } 0 \leq k \leq n - 1, \\
\frac{1 + n}{(1 + r^n)u_n} a_n, & \text{if } k = n, \\
0, & \text{if } k > n.
\end{cases} \]
\[ D_2 = \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k |d_{nk}^r| < \infty \right\}, \]
where, \( D^r = (d_{nk}^r) \) is defined as
\[ d_{nk}^r = \begin{cases} 
(1 + k) \left[ \frac{a_k}{(1 + r^k)u_k} + \sum_{j=0}^{n} \frac{1}{(1 + r^k)u_k} - \frac{1}{(1 + r^{k+1})u_{k+1}} \right] a_n, & \text{if } 0 \leq k \leq n, \\
\frac{n + 1}{(1 + r^n)u_n} a_n, & \text{if } k = n, \\
0, & \text{if } k > n.
\end{cases} \]
\[ D_3 = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} a_j \text{ exists for each fixed } k \in N \right\} \]

and

\[ D_4 = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n-1} \left( \sum_{j=0}^{k} \frac{1}{1 + r^j u_j} \right) a_k \text{ exists } < \infty \right\} . \]

Then,

\[ [a_0^\alpha (u, \Delta)]^\alpha = [a_0^\alpha (u, \Delta)]^\alpha = D_1, [a_0^\gamma (u, \Delta)]^\beta = D_2 \cap D_3, [a^\gamma (u, \Delta)]^\beta = D_2 \cap D_3 \cap D_4 \]

and \([a_0^\gamma (u, \Delta)]^\gamma = a_0^\gamma (u, \Delta)\) = \(D_2\).

**Proof.** Let us take any \( a = (a_k) \in \omega \). We can easily derive with (3) that

\[ a_n x_n = \sum_{k=0}^{n-1} \sum_{j=k-1}^{k} \frac{1 + j}{1 + r^j u_j} y_j = (C^r y)_n, \ n \in \mathbb{N}. \quad (7) \]

Thus, we observe by combining (7) with (i) of Lemma 3.1 that \(ax = (a_n x_n) \in l_1\) whenever \(x = (x_n) \in a_0^\alpha (u, \Delta)\) if and only if \(C^r y \in l_1\) whenever \(y \in c_0\) or \(c\). This gives the result that \([a_0^\alpha (u, \Delta)]^\alpha = [a_0^\gamma (u, \Delta)]^\alpha = D_1\).

Further, consider the equation,

\[ \sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \sum_{j=0}^{k} (-1)^{j-i} \frac{1 + i}{(1 + r^j u_j) y_i} \right] a_k \]

\[ = \sum_{k=0}^{n-1} (1 + k) \left[ \frac{a_k}{1 + r^k u_k} + \left( \frac{1}{1 + r^k u_k} - \frac{1}{(1 + r^{k+1}) u_{k+1}} \right) \sum_{j=k+1}^{n} a_j \right] u_k^{-1} y_k \]

\[ + \frac{n + 1}{(1 + r^n) u_n} a_n y_n. \quad (8) \]
Thus we deduce from Lemma 3.3 with (8) that \( ax = (a_n x_n) \in cs \) whenever \( x = (x_n) \in \alpha^r_c(u, \Delta) \) if and only if \( D_y \in c \) whenever \( y \in c \). Therefore, we derive by using (4), (5) and (6) that

\[ [\alpha^r_0(u, \Delta)]^\beta = D_2 \cap D_3 \cap D_4. \]

Similarly, we can get \([\alpha^r_0(u, \Delta)]^\beta = D_2 \cap D_3\).

Now, as above and by using lemma 3.4 instead of Lemma 3.3 we get \([\alpha^r_0(u, \Delta)]^\gamma = [\alpha^r_0(u, \Delta)]^\gamma = D_2\).

**Theorem 3.6.** Define the sequence \( b^{(k)}(r) = \{b^{(k)}_n(r)\} \) of the elements of the space \( \alpha^r_0(u, \Delta) \) by

\[
\begin{align*}
b^{(k)}_n(r) &= \begin{cases} 
  \frac{1 + k}{(1 + r^k)u_k} - \frac{1 + k}{(1 + r^{k+1})u_{k+1}}, & \text{if } 0 \leq k \leq n - 1, \\
  \frac{1 + n}{(1 + r^n)u_n}, & \text{if } k = n, \\
  0, & \text{if } k > n,
\end{cases}
\end{align*}
\]

for every fixed \( k \in \mathbb{N} \). Then,

(i) The sequence \( \{b^{(k)}(r)\}_{k \in \mathbb{N}} \) is a basis for the space \( \alpha^r_0(u, \Delta) \) and any \( x \in \alpha^r_0(u, \Delta) \) has a unique representation of

\[
x = \sum_k \lambda_k(r)b^{(k)}(r), \tag{9}
\]

(ii) The sequence \( \{t, b^{(k)}(r)\} \) is a basis for the space \( \alpha^r_c(u, \Delta) \) and any \( x \in \alpha^r_c(u, \Delta) \) has a unique representation of

\[
x = lt + \sum_k [\lambda_k(r) - l]b^{(k)}(r), \tag{10}
\]

where,

\[ t_k = \sum_{j=0}^{k} \frac{1}{1 + r^j}, \lambda_k(r) = (B_r x)_k \text{ for all } k \in \mathbb{N} \text{ and } \]
\[ l = \lim_{k \to \infty} (B_x x)_k. \]  

**Proof.** It is clear that \( b^{(k)}(r) \subset a_0^r(u, \Delta) \), since

\[ B_r b^{(k)}(r) = e^{(k)} \in c_0, \tag{12} \]

where \( e^{(k)} \) is the sequence whose only non-zero term is 1 in \( k^{th} \) place for each \( k \in \mathbb{N} \).

Let \( x \in a_0^r(u, \Delta) \) be given. For every non-negative integer \( m \), we put

\[ x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) b^{(k)}(r). \tag{13} \]

Then, we obtain by applying \( B_r \) to (12) with (11) that

\[ B_r x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) B_r b^{(k)}(r) = \sum_{k=0}^{m} (B_r x)_k e^{(k)} \]

and

\[ (B_r (x - x^{[m]}))_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (B_r x)_i, & \text{if } i > m, \end{cases} \]

where \( i, m \in \mathbb{N} \). Given \( \varepsilon > 0 \), there exists an integer \( m_0 \) such that

\[ |(B_r x)_m| < \frac{\varepsilon}{2} \]

for all \( m \geq m_0 \). Hence,

\[ |x - x^{[m]}| = \sup_{n \geq m} |(B_r x)_n| \leq \sup_{n \geq m_0} |(B_r x)_n| < \frac{\varepsilon}{2} < \varepsilon \]

for all \( m \geq m_0 \), which proves that \( x \in a_0^r(u, \Delta) \) is represented as (9).

Let us show the uniqueness of the representation for \( x \in a_0^r(u, \Delta) \) given by (9). Suppose, on the contrary; that there exists a representation \( x = \sum_k \lambda_k(r) b^{k}(r) \). since the linear transformation \( T \) from \( a_0^r(u, \Delta) \) to \( c_0 \) used in the Theorem 2.2 is continuous we have
\[(B_r x)_n = \sum_k \mu_k(r)(B_r b^k(r))_n = \sum_k \mu_k(r)e^{(k)}_n = \mu_n(r)\]

for \(n \in \mathbb{N}\), which contradicts the fact that \((B_r x)_n = \lambda_n(r)\) for all \(n \in \mathbb{N}\). Hence, the representation (9) is unique, which proves first part of the theorem.

(ii) Since, \(\{b^{(k)}(r)\} \subset a^0_c(u, \Delta)\) and \(t \subset a^r_c(u, \Delta)\), the inclusion \(\{t, b^{(k)}(r)\} \subset a^r_c(u, \Delta)\) holds trivially. Let us take \(x \in a^r_c(u, \Delta)\). Then, there unqiely exists an \(l\) satisfying (10). We, therefore, have the fact that \(z \in a^0_c(u, \Delta)\) whenever we set \(z = x - lt\). Therefore, we deduce by the part (i) of the present theorem that the representation of \(z\) is unique. This leads us to the fact that the representation of \(x\) given by (10) is unique and this step completes the proof of the theorem.

4. Matrix Mappings on the Space \(a^r_c(u, \Delta)\)

In this section, we characterize the matrix mappings from the space \(a^r_c(u, \Delta)\) to the spaces \(l_\infty\), and \(c_0\).

For brevity in notation, we shall write

\[\hat{a}_{nk} = (1 + k) \left[ \frac{a_{nk}}{(1 + r^k)u_k} + \left( \frac{1}{(1 + r^k)u_k} - \frac{1}{(r^{k+1})u_{k+1}} \right) \sum_{j=k+1}^\infty a_{nj} \right] \]

for all \(n, k \in \mathbb{N}\).

**Theorem 4.1.** \(A \in (a^r_c(u, \Delta): l_\infty)\) if and only if

\[
\begin{align*}
\left( \frac{(1 + k)a_{nk}}{(1 + r^k)u_k} \right) &\in cs, n \in \mathbb{N}, \\
\sup_{n \in \mathbb{N}} \sum_k |\hat{a}_{nk}| &< \infty.
\end{align*}
\]

Proof. Let \(A \in (a^r_c(u, \Delta): l_\infty)\). Then \(Ax\) exists for \(x \in a^r_c(u, \Delta)\) and implies that \(\{a_{nk}\}_{k \in \mathbb{N}} \in \{a^r_c(u, \Delta)\}^\beta\) for each \(n \in \mathbb{N}\). Hence necessity of
AB. HAMID GANIE and NEYAZ AHMAD SHEIKH

Conversely, suppose that the necessities (14) and (15) hold and take any \( x \in \alpha'_c(u, \Delta) \), since \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\alpha'_c(u, \Delta)\}^\beta \) for every fixed \( n \in \mathbb{N} \), so the \( A \)-transform of \( x \) exists. Consider the following equality obtained by using the relation (3) that

\[
\sum_{k=0}^{m} a_{nk}x_k = (1 + k) \left[ \frac{a_{nk}}{(1 + r^k)u_k} + \left( \frac{1}{(1 + r^k)u_k} - \frac{1}{(r^{k+1})u_{k+1}} \right) \sum_{j=k+1}^{m} a_{nj} \right] y_k
\]

\[+ \frac{1 + m}{(1 + r^m)u_m} a_{mn} y_m. \tag{16}\]

Taking into account the assumptions we derive from (17) as \( m \to \infty \) bearing in mind that second term in (17) tends to zero, that

\[
\sum_k a_{nk} x_k = \sum_k \hat{a}_{nk} y_k, \quad n \in \mathbb{N}, \tag{17}\]

\[\|Ax\|_{l_\infty} = \|Bx\|_{l_\infty} < \infty.\]

This shows that \( Ax \in l_\infty \) whenever \( x \in \alpha'_c(u, \Delta) \). This completes the proof.

**Theorem 4.2.** \( A \in (\alpha'_c(u, \Delta); c) \) if and only if (14), (15) hold and

\[
\lim_{n \to \infty} \hat{a}_{nk} = \alpha_k \quad \text{for each} \quad k \in \mathbb{N}, \tag{18}\]

\[
\lim_{n \to \infty} \hat{a}_{nk} = \alpha. \tag{19}\]

**Proof.** Suppose that \( A \) satisfies the conditions (16)-(19). Let \( x \in \alpha'_c(u, \Delta) \). Then, \( Ax \) exists and it is trivial that the sequence \( y = (y_k) \) connected with the sequence \( x = (x_k) \) by the relation (3) is in \( c \) such that \( y_k \to l \) as \( k \to \infty \). At this stage, we observe from (15) and (18) that

\[
\sum_{j=0}^{k} |\alpha_j| \leq \sup_{n \in \mathbb{N}} \sum_j |\hat{a}_{nj}| < \infty,
\]
holds for every $k \in N$. This leads us to the consequence that $(\alpha_k) \in l_1$.
Considering (17), let us write

$$\sum_k a_{nk} x_k = \sum_k \hat{a}_{nk}(y_k - l) + l\sum_k \hat{a}_{nk}.$$  \hspace{1cm} (20)

In this situation, by letting $n \to \infty$ in (20) we see that the first term on the right tends to $\sum_k \alpha_k(y_k - l)$ by (15) and (19) and the second term tends to $l\alpha$ by (19) and we thus have that

$$(Ax)_n \to \sum_k \alpha_k(y_k - l) + l\alpha,$$

which proves that $A \in (a^r_c(u, \Delta): c)$.

Conversely, we suppose that $A \in (a^r_c(u, \Delta): c)$. Then, since $c \subset l_\infty$
holds, the necessity of (16) and (17) follows immediately from above
Theorem. To prove the necessity of (18), consider the sequence

$x = x^{(k)} = \{x_n^{(k)}\}_{n \in N} \in a^r_c(u, \Delta)$ defined by

$$x_n^{(k)}(r) = \begin{cases} (-1)^{n-k} \frac{1 + k}{(1 + r^k)u_k} & \text{if } k \leq n \leq k + 1, \\ 0, & \text{if } 0 \leq n \leq k - 1 \text{ or } n > k + 1, \end{cases}$$

for each $k \in N$. Since $Ax$ exists and is in $c$ for every $x \in a^r_c(u, \Delta)$, we can
easily see that $Ax^k = \{\hat{a}_{nk}\}_{n \in N} \in c$ for each $k \in N$ which shows the
necessity of (18). Similarly, by taking $x = e$ in (17), we also obtain that

$Ax = \{\sum_k \hat{a}_{nk}\}_{n \in N}$ which belongs to the space $c$ and this shows the
necessity of (19), hence the proof of the theorem is complete.

References


