APPLICATION OF HOMOTOPY ANALYSIS METHOD FOR SOLVING A CLASS OF NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

SH. SADIGH BEHZADI
Department of Mathematics
Islamic Azad University
Branch Qazvin, Qazvin
Iran
e-mail: shadan_Behzadi@yahoo.com

Abstract
In this paper, the nonlinear Volterra-Fredholm integro-differential equations are solved by using the homotopy analysis method (HAM). The approximation solution of this equation is calculated in the form of a series which its components are computed easily. The existence and uniqueness of the solution and the convergence of the proposed method are proved. A numerical example is studied to demonstrate the accuracy of the presented method.

1. Introduction
Since many physical problems are modeled by integro-differential equations, the numerical solutions of such integro-differential equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations and integro-differential equations for example [1-5, 15, 16, 18], The homotopy analysis method (HAM) is based on homotopy, a fundamental concept

2010 Mathematics Subject Classification: kindely provide.
Keywords and phrases: volterra and fredholm integral equations, integro-differential equations, homotopy analysis method (HAM).
Received October 12, 2011
in topology and differential geometry [12], The HAM has successfully been applied to many situations [1, 6-14, 19-21]. In [17] we had studied the high-order nonlinear Volterra-Fredholm integro-differential equation by using HAM of the form

$$\sum_{j=0}^{K} P_j(x) y^{(j)}(x) = f(x) + \mu_1 \int_{a}^{x} k_1(x, t) G_1(y^{(p)}(t)) \, dt$$

$$+ \mu_2 \int_{a}^{b} k_2(x, t) G_2(y^{(m)}(t)) \, dt, \quad 0 \leq p, m \leq k.$$ 

In this study, we develop HAM to solve the high-order nonlinear Volterra-Fredholm integro-differential equations as follows:

$$\sum_{j=0}^{k} p_j(x) y^{(j)}(x) = f(x) + \mu_1 \int_{a}^{x} k_1(x, t) G_1(y^{(p)}(t)) \, dt$$

$$+ \mu_2 \int_{a}^{b} k_2(x, t) G_2(y^{(m)}(t)) \, dt, \quad 0 \leq p, \ m \leq k. \quad (1)$$

With initial conditions

$$y^{(r)}(a) = b_r, \quad r = 0, 1, 2, ..., k - 1,$$ 

where $a, b, \mu_1, \mu_2, b_r$ are constant values, $f(x), k_1(x, t), k_2(x, t), G_1(y(t)), G_2(y(t))$ and $p_j(x), j = 0, 1, ..., k$ are functions that have suitable derivatives on an interval $a \leq t \leq x \leq b$ and $p_k(x) \neq 0$.

The paper is organized as follows. In section 2, the HAM is briefly presented, in section 3, this method is presented for solving Eq. (3). Also, the existence and uniqueness of the solution and convergence of the proposed method are proved. Finally, a numerical example and computational complexity of the proposed algorithm are shown in section 4.

2. Preliminaries

Consider

$$N(y) = 0,$$
where \( N \) is a nonlinear operator, \( y(x) \) is unknown function and \( x \) is an independent variable, let \( y_0(x) \) denote an initial guess of the exact solution \( y(x) \), \( h \neq 0 \) an auxiliary parameter, \( H(x) \neq 0 \) an auxiliary function, and \( L \) an auxiliary linear operator with the property \( L(r(x)) = 0 \) when \( r(x) = 0 \). Then using \( q \in [0, 1] \) as an embedding parameter, we construct a homotopy as follows.

\[
(1 - q) L \left[ \phi (x; q) - y_0(x) \right] - q h H (x) N \left[ \phi (x; q) \right] = \hat{H} \left[ \phi (x; q) ; y_0(x), H(x), h, q \right].
\]  

(3)

It should be emphasized that we have great freedom to choose the initial guess \( y_0(x) \), the auxiliary linear operator \( L \), the non-zero auxiliary parameter \( h \), and the auxiliary function \( H(x) \).

Enforcing the homotopy (3) to be zero, i.e.,

\[
\hat{H}[\phi (x; q) ; y_0(x), H(x), h, q] = 0,
\]  

(4)

We have so-called zero-order deformation equation

\[
(1 - q)L [\phi (x; q) - y_0(x)] = q h H (x) N[\phi (x; q)].
\]  

(5)

When \( q = 0 \), the zero-order deformation Eq. (5) becomes

\[
\phi (x; 0) = y_0(x),
\]  

(6)

and when \( q = 1 \), since \( h \neq 0 \) and \( H(x) \neq 0 \), the Eq. (5) is equivalent to

\[
\phi (x; 1) = y(x),
\]  

(7)

Thus, according to (6) and (7), as the embedding parameter \( q \) increases from 0 to 1, \( \phi (x; q) \) varies continuously from the initial approximation \( y_0(x) \) to the exact solution \( y(x) \). Such a kind of continuous variation is called deformation in homotopy [6].

Due to Taylor’s theorem, \( \phi (x; q) \) can be expanded in a power series of \( q \) as follows

\[
\phi (x; q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m,
\]  

(8)
Where,

\[ y_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} |_{q=0}. \]

Let the initial guess \( y_0(x) \), the auxiliary linear parameter \( L \), the nonzero auxiliary parameter \( h \) and the auxiliary function \( H(x) \) be properly chosen so that the power series (8) of \( \phi(x; q) \) converges at \( q = 1 \), then, we have under these assumptions the solution series

\[ y(x) = \phi(x; 1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \]  

(9)

From Eq. 8, we can write Eq. 5, as follows

\[ L \left[ \sum_{m=1}^{\infty} y_m(x) q^m \right] - q L \left[ \sum_{m=1}^{\infty} y_m(x) q^m \right] = qhH(x)N[\phi(x, q)]. \]  

(10)

By differentiating (10) \( m \) times with respect to \( q \), we obtain

\[ m! L[y_m(x) - y_{m-1}(x)] = h H(x) m \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^{m-1}} |_{q=0}. \]

Therefore,

\[ L[y_m(x) - \chi_m y_{m-1}(x)] = h H(x) \mathcal{R}_m(y_{m-1}(x)), \quad y_m(0) = 0, \]  

(11)

Where,

\[ \mathcal{R}_m(y_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^{m-1}} |_{q=0}, \]  

(12)

and

\[ \chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m < 1. \end{cases} \]

Note that the high-order deformation Eq. (11) is governing the linear operator \( L \), and the term \( \mathcal{R}_m(y_{m-1}(x)) \) can be expressed simply by (12) for any nonlinear operator \( N \).
3. Description of the Method

To obtain the approximation solution of Eq. (2) based on the HAM let

\[ N(y) = y(x) - L^{-1} \left( \frac{f(x)}{p_h(x)} \right) \]

\[ - \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^r b_r - \mu_1 L^{-1} \left( \int_a^x \frac{k_1(x, t)}{p_h(x)} G_1(y^{(p)}(t)) \, dt \right) \]

\[ - \mu_2 L^{-1} \left( \int_a^b \frac{k_2(x, t)}{p_h(x)} G_2(y^{(m)}(t)) \, dt \right) + L^{-1} \left( \sum_{j=0}^{k-1} \frac{p_j(x)}{p_h(x)} y^{(j)}(x) \right) = 0, \]

where \( L^{-1} \) is the multiple integration operators as follows.

\[ L^{-1}(\cdot) = \int_a^x \int_a^x \int_a^x \cdots \int_a^x (\cdot) \, dx \cdots \, dx, \quad (k \text{ times}). \]

We obtain the term \( \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^r b_r \) from the initial conditions.

We have

\[ \mathcal{G}_m(y_{m-1}(x)) = y_{m-1}(x) - \mu_1 L^{-1} \left( \int_a^x \frac{k_1(x, t)}{p_h(x)} G_1(y^{(p)}_{m-1}(t)) \, dt \right) \]

\[ - \mu_2 L^{-1} \left( \int_a^b \frac{k_2(x, t)}{p_h(x)} G_2(y^{(m)}_{m-1}(t)) \, dt \right) \]

\[ + L^{-1} \left( \sum_{j=0}^{K-1} \frac{p_j(x)}{p_h(x)} y^{(j)}_{m-1}(x) \right) - (1 - \chi_m) \]

\[ \left( L^{-1} \left( \frac{f(x)}{p_h(x)} \right) + \sum_{r=0}^{k-1} \frac{1}{r!} (x-a)^r b_r \right), \quad m \geq 1. \]

Substituting (13) into (11),

\[ L \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = \mu_1 L^{-1} \left( \int_a^x \frac{k_1(x, t)}{p_h(x)} G_1(y^{(p)}_{m-1}(t)) \, dt \right) \]

\[ - \mu_2 L^{-1} \left( \int_a^b \frac{k_2(x, t)}{p_h(x)} G_2(y^{(m)}_{m-1}(t)) \, dt \right) - \mu_2 L^{-1} \left( \int_a^b \frac{k_2(x, t)}{p_h(x)} G_2(y^{(m)}_{m-1}(t)) \, dt \right) \]
We take an initial guess $y_0(x) = F(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right)$

$$+ \sum_{r=0}^{k-1} \frac{1}{r!} (x - a)^r b_r,$$

an auxiliary nonlinear operator $L y = y$, a nonzero auxiliary parameter $h = -1$, and auxiliary function $H(x) = 1$.

This is substituted into (14) to give the recurrence relation

$$y_0(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!} (x - a)^r b_r,$$

$$y_m(x) = \mu_1 L^{-1}\left(\int_a^x \frac{k_1(x, t)}{p_k(x)} G_1(y^{(p)}(t)) dt\right) + \mu_2 L^{-1}\left(\int_a^b \frac{k_2(x, t)}{p_k(x)} G_2(y^{(m)}(t)) dt\right)$$

$$G_2(y^{(m)}(t)) dt - L^{-1}\left(\sum_{j=0}^{k-1} \frac{P_j(x)}{p_k(x)} y^{(j)}(x)\right), \quad m \geq 1. \quad (15)$$

Relation (15) will enable us to determine the components $y_m(x)$ recursively for $m \geq 0$.

Since $p_k(x) \neq 0$, we can write Eq. (2) in the form

$$y(x) = L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!} (x - a)^r b_r$$

$$+ \mu_1 L^{-1}\left(\int_a^x \frac{k_1(x, t)}{p_k(x)} G_1(y^{(p)}(t)) dt\right)$$

$$+ \mu_2 L^{-1}\left(\int_a^b \frac{k_2(x, t)}{p_k(x)} G_2(y^{(m)}(t)) dt\right)$$

$$- L^{-1}\left(\sum_{j=0}^{k-1} \frac{P_j(x)}{p_k(x)} y^{(j)}(x)\right). \quad (16)$$

The following relations have been mentioned in [16]:
\[ L^{-1}\left(\int_a^x \frac{k_1(x, t)}{p_k(x)} G_1(y^{(p)}(t)) \, dt\right) = \frac{1}{k!} \int_a^x (x - t)^k \frac{k_1(x, t)}{p_k(x)} G_1(y^{(p)}(t)) \, dt. \quad (17) \]

\[ \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} \right) y^{(j)}(t) = \sum_{j=0}^{k-1} \frac{1}{k!} \int_a^x (x - t)^k \frac{p_j(x)}{p_k(x)} y^{(j)}(t) \, dt. \quad (18) \]

By substituting (17) and (18) in (16) we obtain
\[
y(x) = F(x) + \mu_1 \int_a^x L^{-1} \left( \frac{k_2(x, t)}{p_k(x)} \right) G_2(y^{(m)}(t)) \, dt
\]
\[
+ \frac{\mu_1}{k!} \int_a^x (x - t)^k \frac{k_1(x, t)}{p_k(x)} G_1(y^{(r)}(t)) \, dt
\]
\[
- \sum_{j=0}^{k-1} \frac{1}{k!} \int_a^x (x - t)^k \frac{p_j(x)}{p_k(x)} y^{(j)}(t) \, dt. \quad (19) \]

In (19), we assume that \( F(x) \) is bounded for all \( t \) in \( C = [a, b] \) and
\[
\left| \frac{\mu_1 k_1(x, t)(x - t)^k}{k! p_k(x)} \right| \leq M', \quad \left| \mu_2 L^{-1} \left( \frac{k_2(x, t)}{p_k(x)} \right) \right| \leq M''
\]
\[
\left| \frac{(x - t)^k p_j(x)}{p_k(x) k!} \right| \leq M_j, \quad j = 0, 1, ..., k - 1, \quad \forall t \leq x \leq b.
\]

Also, we suppose the nonlinear terms \( G_1(y^{(p)}(t)), G_2(y^{(m)}(t)) \) and \( D^j y(t) = \frac{d^j}{dt^j} y(t) \) are Lipschitz continuous with
\[
G_1(y^{(p)}) - G_1(y^{(p)}) \leq L, \quad y^{(p)} - y^{*(p)} \leq L, \quad G_2(y^{(m)}) - G_2(y^{(m)}) \leq L, \quad y^{(m)} - y^{*(m)} \leq L, \quad D^j y - D^j y^* \leq L_j \quad \forall j = 0, 1, ..., k - 1.
\]

If we set
\[
\alpha = (L'M' + L'M' + kLM) (b - a), \quad M = \max |M_j|, \quad L
\]
\[
= \max |L_j|, \quad j = 0, 1, ..., k - 1,
\]

Then the following theorems can be proved by using the above assumptions.
Theorem 3.1. The nonlinear Volterra-Fredholm integro-differential equation in Eq. (2), has a unique solution whenever \( 0 < \alpha < 1 \).

**Proof.** Let \( y \) and \( y^* \) be two different solutions of (19) then

\[
|y - y^*| = \left| \int_a^x \frac{\mu_1 k_1(x, t)(x - t)^h}{p_k(x)k!} [G_1(y) - G_1(y^*)] \, dt \right|
\]

\[
+ \int_a^b \mu_2 L^{-1}\left( \frac{k_2(x, t)}{p_k(x)} \right) \left( |G_2(y) - G_2(y^*)| \right) dt
\]

\[
- \sum_{j=0}^{b-1} \int_a^x \left| (x - t)^j \frac{P_j(x)}{p_k(x)k!} \right| \left| D^j(y) - D^j(y^*) \right| \, dt
\]

\[
\leq \left| \int_a^x \frac{\mu_1 k_1(x, t)(x - t)^h}{p_k(x)k!} \left| G_1(y) - G_1(y^*) \right| \, dt \right|
\]

\[
+ \int_a^b \mu_2 L^{-1}\left( \frac{k_2(x, t)}{p_k(x)} \right) \left( |G_2(y) - G_2(y^*)| \right) dt
\]

\[
+ \sum_{j=0}^{b-1} \int_a^x \left| (x - t)^j \frac{P_j(x)}{p_k(x)k!} \right| \left| D^j(y) - D^j(y^*) \right| \, dt
\]

\[
\leq (b - a)(L^M + \bar{L}M^\nu + kLM) |y - y^*|,
\]

From which we get \((1 - \alpha)|y - y^*| \leq 0\). Since \( 0 < \alpha < 1 \), so \(|y - y^*| = 0\). Therefore, \( y = u^* \) and this completes the proof.

Theorem 3.2. If the series solution \( y(x) = \sum_{m=0}^{\infty} y_m(x) \) obtained from (15) is convergent then it converges to the exact solution of the problem (2).

**Proof.** We assume:

\[
y^{(j)}(x) = \sum_{m=0}^{\infty} y^{(j)}_m(x),
\]

\[
S_1(y^{(p)}(x)) = \sum_{m=0}^{\infty} G_1(y^{(p)}_m(x)),
\]
APPLICATION OF HOMOTOPY ANALYSIS METHOD ...

\[ S_2(y^{(m)}(x)) = \sum_{m=0}^{\infty} G_2(y^m_m(x)), \]  
(20)

where,

\[ \lim_{m \to \infty} y_m(x) = 0. \]

We can write

\[ \sum_{m=1}^{n} \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = y_1 + (y_2 - y_1) + \ldots + (y_n - y_{n-1}) = y_n(x). \]  
(21)

Hence, from (21),

\[ \lim_{n \to \infty} y_n(x) = 0. \]  
(22)

So, using (22) and the definition of the linear operator \( L \), we have

\[ \sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = L \left[ \sum_{m=1}^{\infty} [y_m(x) - \chi_m y_{m-1}(x)] \right] = 0. \]

Therefore, from (11) we can obtain that

\[ \sum_{m=1}^{\infty} L \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = hH(x) \sum_{m=1}^{\infty} \Re_m(y_{m-1}(x)) = 0. \]

Since \( h \neq 0 \) and \( H(x) \neq 0 \), we have

\[ \sum_{m=1}^{\infty} \Re_m(y_{m-1}(x)) = 0. \]  
(23)

By applying the relations (13) and (20),

\[ \sum_{m=1}^{\infty} \Re_m(y_{m-1}(x)) = \sum_{m=1}^{\infty} \left[ y_{m-1} - \mu_1 L^{-1} \left[ \int_a^b k_1(x, t) \frac{P_k(x)}{P_k(t)} y_{m-1}(t) dt \right] - \mu_2 L^{-1} \left[ \int_a^b k_2(x, t) \frac{P_k(x)}{P_k(t)} y_{m-1}(t) dt \right] \right. \]
\[ \left. \left. + L^{-1} \left[ \sum_{j=0}^{k-1} \frac{P_j(x)}{P_k(x)} \frac{P_{j+1}(x)}{P_k(x)} y_{m-1}(x) \right] (1 - \chi_n) F(x) \right] \right] = y(x) - F(x) - \mu_1 L^{-1} \left[ \int_a^b k_1(x, t) \frac{P_k(x)}{P_k(t)} \sum_{m=1}^{\infty} \left[ y_{m-1}(t) \right] \right] - \mu_2 L^{-1} \left[ \int_a^b k_2(x, t) \frac{P_k(x)}{P_k(t)} \right] \]

\[ = 0. \]
\[
\left[ \sum_{m=1}^{\infty} G_2(y_m^{(m)}(t)) \right] dt + L^{-1} \left( \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} \sum_{m=1}^{\infty} y_m^{(j)}(x) \right) = y(x) - F(x) \\
- \mu_1 L^{-1} \left( \int_a^x \frac{k_1(x, t)}{p_k(x)} S_1(y^{(p)}(t)) \, dt \right) - \mu_2 L^{-1} \left( \int_a^x \frac{k_2(x, t)}{p_k(x)} S_2(y^{(m)}(t)) \, dt \right)
\]

\[
L^{-1} \left( \int_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} y^{(j)}(x) \right).
\]

(24)

From (23) and (24), we have

\[
y(x) = F(x) + \mu_1 L^{-1} \left( \int_a^x \frac{k_1(x, t)}{p_k(x)} S_1(y^{(p)}(t)) \, dt \right) \\
+ \mu_2 L^{-1} \left( \int_a^x \frac{k_2(x, t)}{p_k(x)} S_2(y^{(m)}(t)) \, dt \right) - L^{-1} \left( \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} y^{(j)}(x) \right),
\]

Therefore, \( y(x) \) must be the exact solution of Eq. (2).

4. Numerical Examples

In this section, we compute a numerical example which is solved by the HAM. The programs have been provided with mathematica 6 according to the following algorithms where \( \varepsilon \) is a given positive value.

**Algorithm:**

Step 1. \( n \leftarrow 0 \),

Step 2. Calculate the recursive relation using (15),

Step 3. If \( |y_{n+1} - y_n| < \varepsilon \) then go to step 4 else \( n \leftarrow n + 1 \) and go to step 2,

Step 4. Print \( y(x) = \sum_{i=0}^{n} y_i \) as the approximate of the exact solution.

**Lemma 1.** The computational complexity of the algorithm is \( O(k^2 + n) \).
Proof. The number of computations including division, production, sum and subtraction and without considering the number of computations in the term \( \sum_{j=0}^{k-1} L^{-1} \left( \frac{p_j(x)}{p_k(x)} y_n^{(j)}(x) \right), n \geq 1 \).

In step 2,

\[
y_0 : \frac{k^2}{2} + \frac{9}{2} k + 2,
\]

\[
y_1 : 7,
\]

\[
\vdots
\]

\[
y_{n+1} : 7, n \geq 0.
\]

In step 4, the total number of the computations is equal to

\[
y_0 + \sum_{i=1}^{n} y_i = 7n + \frac{k^2}{2} + \frac{9}{2} K + 2 = O(k^2 + n).
\]

Example 4.1. Let us now study the nonlinear integro-differential equation

\[
u''(x) + xu'(x) = e^x (2 + x^2 + 3x) - (0.5892858)x
\]

\[
+ \int_0^x [u(t)]^2 \, dt + \int_0^{0.5} x(t1 + u(t))^2 \, dt,
\]

With initial conditions \( u(0) = 0, u'(0) = 1 \). The exact solution is \( u(x) = xe^x \cdot e = 10^{-3} \) and \( \alpha = 0.350099 \).

5. Conclusion

Homotopy analysis method has been known as a powerful scheme for solving many functional equations such as algebraic equations, ordinary and partial differential equations, integral equations and so on. The HAM has been shown to solve effectively, easily and accurately a large class of nonlinear problems with the approximations which are rapidly convergent to the exact solution. In this work, the HAM has been successfully employed to obtain the approximate or analytical solution of the nonlinear Volterra-Fredholm integro-differential equations.
Table 1. Numerical results for Example 1

<table>
<thead>
<tr>
<th>$x$</th>
<th>App. Sol($n=3$)</th>
<th>Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.0528836</td>
<td>0.00071584</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1118610</td>
<td>0.00127473</td>
</tr>
<tr>
<td>0.12</td>
<td>0.1372840</td>
<td>0.00146522</td>
</tr>
<tr>
<td>0.15</td>
<td>0.1775160</td>
<td>0.00163066</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2505650</td>
<td>0.00196136</td>
</tr>
<tr>
<td>0.25</td>
<td>0.3318580</td>
<td>0.00217893</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4223900</td>
<td>0.00235548</td>
</tr>
<tr>
<td>0.35</td>
<td>0.5233110</td>
<td>0.00268905</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6359360</td>
<td>0.00297645</td>
</tr>
<tr>
<td>0.45</td>
<td>0.7617640</td>
<td>0.00314973</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9024930</td>
<td>0.00347981</td>
</tr>
</tbody>
</table>

Table 1 shows that, the approximation solution of the nonlinear Fredholm-Volterra integral equation is convergent with 3 iterations by using the HAM.

References


APPLICATION OF HOMOTOPY ANALYSIS METHOD ...


